Chapter 2

Partial derivatives

2.1 Introduction

Before we actually start taking derivatives of functions of more than one variable let's recall an important interpretation of derivatives of functions of one variable. Recall that given a function of one variable, $f(x)$, the derivative, $f'(x)$ represents the rate of change of the function as x changes. This is an important interpretation of derivatives and we are not going to want to lose it with functions of more than one variable. The problem with functions of more than one variable is that there is more than one variable. In other words, what do we do if we only want one of the variables to change, or if we want more than one of them to change? In fact, if we're going to allow more than one of the variables to change there are then going to be an infinite amount of ways for them to change. For instance, one variable could be changing faster than the other variables in the function. Notice as well that it will be completely possible for the function to be changing differently depending on how we allow one or more of the variables to change. We will need to develop ways, and notations, for dealing with all of these cases. In this section

we are going to concentrate exclusively on only changing one of the variables at a time, while the remaining variables are held fixed. We will deal with allowing multiple variables to change in a later section. Because we are going to only allow one of the variables to change taking the derivative will now become a fairly simple process. Let's start off this discussion with a fairly simple function. Let's start with the function $f(x,y) = 2x^2y^3$ and let's determine the rate at which the function is changing at a point(a,b), if we hold y fixed and allow x to vary and if we hold x fixed and allow y to vary.

We'll start by looking at the case of holding y fixed and allowing x to vary. Since we are interested in the rate of change of the function at (a,b) and are holding y fixed this means that we are going to always have y = b (if we didn't have this then eventually y would have to change in order to get to the point…). Doing this will give us a function involving only x's and we can define a new function as follows, $g(x) = f(x,b) = 2x^2b^3$.

Now, this is a function of a single variable and at this point all that we are asking is to determine the rate of change of $g(x)$ at $x = a$. In other words, we want to compute $g'(a)$ and since this is a function of a single variable we already know how to do that. Here is the rate of change of the function at (a,b) if we hold y fixed and allow x to vary such that $g'(a) = 4ab^3$. We will call $g'(a)$ the partial derivative of $f(x,y)$ with respect to x at (a,b) and we will denote it in the following way, $f_x(a,b) = 4ab^3$. Now, let's do it the other way. We will now hold x fixed and allow y to vary. We can do this in a similar way. Since we are holding x fixed it must be fixed at $x = a$ and so we can define a new function of y and then differentiate this as we've always done with functions of one variable. Here is the work for this, $h(y) = f(a,y) = 2a^2 y^3$, $h'(b) = 6a^2 b^2$. In this case we call $h'(b)$ the partial derivative of $f(x,y)$ with respect to y at (a,b) and we denote it as follows, $f_y(a,b) = 6a^2 b^2$.

Note that these two partial derivatives are sometimes called the first order partial derivatives. Just as with functions of

one variable we can have derivatives of all orders. We will be looking at higher order derivatives in a later section. Note that the notation for partial derivatives is different than that for derivatives of functions of a single variable. With functions of a single variable we could denote the derivative with a single prime. However, with partial derivatives we will always need to remember the variable that we are differentiating with respect to and so we will subscript the variable that we differentiated with respect to. We will shortly be seeing some alternate notation for partial derivatives as well. Note as well that we usually don't use the (a,b) notation for partial derivatives. The more standard notation is to just continue to use (x,y). So, the partial derivatives from above will more commonly be written as,

$f_x(x,y) = 4xy^3$ and $f_y(x,y) = 6x^2y^2$. Now, as this quick example has shown taking derivatives of functions of more than one variable is done in pretty much the same manner as taking derivatives of a single variable. To compute $f_x(x,y)$ all we need to do is treat all the y's as constants (or numbers) and then differentiate the x's as we've always done.

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Likewise, to compute $f_y(x,y)$, we will treat all the x's as constants and then differentiate the y's as we are used to doing. Before we work any examples let's get the formal definition of the partial derivative out of the way as well as some alternate notation. Since we can think of the two partial derivatives above as derivatives of single variable functions it shouldn't be too surprising that the definition of each is very similar to the definition of the derivative for single variable functions. Here are the formal definitions of the two definitions partial derivatives we looked at above.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x,y+k)-f(x,y)}{k} \qquad (1)$$

Now let's take a quick look at some of the possible alternate notations for partial derivative. Given the function $z = f(x,y)$ the following are all equivalent notations, $f_x (x,y) = f_x =$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = z_x = \frac{\partial z}{\partial x} \ \& \ f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = z_y = \frac{\partial z}{\partial y}.$$

For the fractional notation for the partial derivative notice the difference between the partial derivative and the ordinary derivative from single variable calculus.

$$f(x) \Rightarrow f'(x) = \frac{df}{dx}, \quad f(x,y) \Rightarrow f_x(x,y) = \frac{\partial f}{\partial x} \text{ and } f_y(x,y) = \frac{\partial f}{\partial y}$$

Now let's work some examples. When working these examples always keep in mind that we need to pay very close attention to which variable we are differentiating with respect to. This is important because we are going to treat all other variables as constants and then proceed with the derivative as if it was a function of a single variable. If you can remember this you'll find that doing partial derivatives are not much more difficult that doing derivatives in of functions of a single variable as we did in Calculus I.

Let $f(x,y)$ be a function with two variables. If we keep y constant and differentiate f (assuming f is differentiable) with respect to the variable x, we obtain what is called the partial derivative of f with respect to x which is denoted by $\frac{\partial f}{\partial x}$ or f_x . Similarly If we keep x constant and differentiate f (assuming f is differentiable) with respect to the variable y, we obtain what is called the partial derivative of f with respect to y which is denoted by $\frac{\partial f}{\partial y}$ or f_y. We now present several

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examples with detailed solution on how to calculate partial derivatives.

Example 1

Find the partial derivatives f_x and f_y if $f(x , y)$ is given by

a) $f(x,y) = x^2y + 2x + y$

b) $f(x,y) = y^7\ln x + \dfrac{9}{y^3} + \sqrt[7]{x^4}$

Solution:

Assume y is constant and differentiate with respect to x to obtain

a) $f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}(x^2y + 2x + y) = 2xy+2x,$

b) $f_x = \dfrac{\partial f}{\partial x} = \dfrac{y^7}{x} + \dfrac{4}{7\sqrt[7]{x^3}}$

Then assume x is constant and differentiate with respect to y to obtain

a) $f_y = \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}(x^2y + 2x + y) = x^2 + 1,$

b) $f_y = \dfrac{\partial f}{\partial y} = 7y^6 \ln x - \dfrac{27}{y^4}$

Example 2

Find all of the first order partial derivatives for the following functions.

a) $w(x,y,z) = x^2y - 10\, y^2z^2 + 43x - 7\tan(4y),$

b) $f(x,y) = \cos(\dfrac{4}{x})\, e^{x^2y-5y^3},$

Solution:

a) $w_x = 2xy + 43,\quad w_y = x^2 - 20\, yz^2 - 28\sec^2(4y),\quad w_z = -20\, y^2z$

b) $f_x = -\dfrac{4}{x^2}\sin(\dfrac{4}{x})e^{x^2y-5y^3} - 2xy\cos(\dfrac{4}{x})e^{x^2y-5y^3},$

$f_y = (x^2y - 15y^2)\cos(\dfrac{4}{x})\, e^{x^2y-5y^3},$

2.2 Total derivative

If the function is implicit such that $f(x,y) = 0$, then the first derivative y' can be calculated as $f_x(x,y) + f_y(x,y)\ y' = 0$, hence

$$y' = -\frac{f_x}{f_y} \tag{2}$$

From second derivative, we will get:

$$f_{xx} + f_{xy}\ y' + f_{yy}\ (y')^2 + f_{yx}\ y' + f_y\ y'' = 0,$$

therefore

$$f_{xx} + 2f_{xy}\ \left(-\frac{f_x}{f_y}\right) + f_{yy}\ \left(-\frac{f_x}{f_y}\right)^2 + f_y\ y'' = 0,$$

Thus

$$y'' = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3} \tag{3}$$

Example 3

Find the first and second derivatives for the function

$x\cos(xy) + e^{xy} = 0$

Solution:

Let $f(x,y) = x\cos(xy) + e^{xy}$, $f_x = \cos(xy) - xy\sin(xy) + ye^{xy}$,

and $f_y = -x^2 \sin(xy) + xe^{xy}$, thus $f_{xx} = -2y\sin(xy) - xy^2\cos(xy)$

$+ y^2e^{xy}$, and $f_{yy} = -x^3 \cos(xy) + x^2e^{xy}$, $f_{xy} = -2x \sin(xy) - x^2y$

$\cos(xy) + e^{xy} + xye^{xy}$.

Therefore

$$y' = -\frac{f_x}{f_y} = \frac{\cos(xy) - xy\sin(xy) + ye^{xy}}{-x^2 \sin(xy) + xe^{xy}}$$

Hence

$$y'' = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}$$

2.3 Applications on partial derivatives

Taylor expansion

To expand the function in two variables $f(x,y)$ about the point (a,b) using Taylor expansion, we have to get f_x, f_y, f_{xx}, f_{xy}, f_{yy}, then substitute in the following expansion at (a,b) such that: $f(x,y) = f(a,b) + \frac{1}{1!}[f_x(a,b)\ (x-a) + f_y(a,b)(y-b)]$

$+ \frac{1}{2!}[f_{xx}(a,b)(x-a)^2 + 2(x-a)(y-b)f_{xy}(a,b) + f_{yy}(a,b)(y-b) \qquad (4)$

Example 4

Expand the function $f(x,y) = e^{xy}\cos(x+y)$ about $(0, \pi)$ using Taylor expansion.

Solution:

We have to get f_x, f_y, f_{xx}, f_{xy}, f_{yy} such that:

$f_x = ye^{xy}\cos(x+y) - e^{xy}\sin(x+y), f_y = xe^{xy}\cos(x+y) - e^{xy}\sin(x+y),$

$f_{xx} = y\ e^{xy}\ (y\cos(x+y) - \sin(x+y)) + e^{xy}(-y\sin(x+y) - \cos(x+y)),$

$f_{yy} = x\ e^{xy}(x\cos(x+y) - \sin(x+y)) + e^{xy}(-x\sin(x+y) - \cos(x+y)),$

$f_{xy} = x\ e^{xy}(y\cos(x+y) - \sin(x+y)) + e^{xy}(-y\sin(x+y)).$

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Therefore: at $(0, \pi)$

$f(0, \pi) = -1$, $f_x(0, \pi) = -\pi$, $f_y(0, \pi) = 0$, $f_{xx} = -\pi^2 + 1$, $f_{yy} = 1$, $f_{xy} = 0$.

Substitute in (4), thus $f(x,y) = f(0,\pi) + \dfrac{1}{1!}[f_x(0,\pi)\ (x\text{-}0) +$

$f_y(0,\pi)(y\text{-}\pi)] + \dfrac{1}{2!}\ [\ f_{xx}(0,\pi)\ (x\text{-}0)^2 + 2(x\text{-}a)(y\text{-}\pi)\ f_{xy}(0,\pi) +$

$f_{yy}(0,\pi)\ (y\text{-}\pi)^2)$, therefore

$$f(x,y) = -1 + \frac{1}{1!}\ (-\pi x\ ) + \frac{1}{2!}\ ((-\pi^2 + 1)\ x^2 + (y\text{-}\pi)^2)$$

Example 5

Expand the function $f(x,y) = xy\ e^{xy}$ about $(1, 0)$ using Taylor expansion.

Solution:

We have to get f_x, f_y, f_{xx}, f_{xy}, f_{yy} such that:

$f_x = ye^{xy}(xy + 1)$, $f_y = xe^{xy}(xy + 1)$, $f_{xx} = y^2 e^{xy}\ (xy + 2)$,

$f_{yy} = x^2 e^{xy}\ (xy + 2)$, and $f_{xy} = e^{xy}\ (xy + 1)^2 + xy\ e^{xy}$.

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Therefore: at (1, 0)

$f(1, 0) = 0, f_x = 0, f_y = 1, f_{xx} = 0, f_{yy} = 2, f_{xy} = 1$

Substitute in (4), we will get :

$$f(x,y) = f(1,0) + \frac{1}{1!} (f_x(1,0)\ (x-1) + f_y(1,0)\ (y-0)) +$$

$$\frac{1}{2!} (\ f_{xx}(1,0)\ (x-1)^2 + 2(x-1)\ (y-0)\ f_{xy}(1, 0) + f_{yy}(1, 0)\ (y-0)^2),$$

thus $f(x,y) = y + (x-1)\ y + y^2$.

Problems

Expand in Taylor the following functions

1) $f(x,y) = e^{iy} \sin x$ about $(\frac{\pi}{2},0)$

2) $f(x,y) = \sin(e^{ixy})$ about $(\frac{\pi}{2},0)$

3) $f(x,y) = x^2 \sin xy$ about $(1,0)$

4) $f(x,y) = e^x \cos y$ about $(1,\frac{\pi}{4})$

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Taylor Maclaurin expansion

If the above point is (0,0), i.e. a = b = 0, then the above expansion is called Taylor Maclaurin so that: $f(x,y) = f(0,0)$

$+ \dfrac{1}{1!} (x\, f_x(0,0) + y\, f_y(0,0)) + \dfrac{1}{2!} (x^2\, f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2\, f_{yy}(0,0))$.

Example 6

Expand the function $f(x,y) = \ln(x+y+1)$ using Taylor Maclaurin expansion.

Solution:

We have to get f_x, f_y, f_{xx}, f_{xy}, f_{yy} such that $f_x = \dfrac{1}{x+y+1}$,

$f_y = \dfrac{1}{x+y+1}$, $f_{xx} = \dfrac{-1}{(x+y+1)^2}$, $f_{yy} = \dfrac{-1}{(x+y+1)^2}$,

$f_{xy} = \dfrac{-1}{(x+y+1)^2}$.

Therefore: at (0, 0)

$f(0,0) = 0$, $f_x = 1$, $f_y = 1$, $f_{xx} = -1$, $f_{yy} = -1$, $f_{xy} = -1$

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Substitute in (4), we will get :

$$f(x,y)= f(0,0) + \frac{1}{1!} (f_x(0,0) (x-0) + f_y(0,0) (y-0)) +$$

$$\frac{1}{2!} (f_{xx}(0,0)(x-0)^2 + 2(x-0) (y-0) f_{xy}(0, 0) + f_{yy}(0, 0) (y-0)^2) ,$$

therefore $\quad f(x,y) = x + y - \frac{1}{2!} [x^2 + 2x y + y^2]$

Envelope

In geometry, an envelope of a family of curves in the plane is a curve that is tangent to each member of the family at some point. Classically, a point on the envelope can be thought of as the intersection of two "adjacent" curves, meaning the limit of intersections of nearby curves. This idea can be generalized to an envelop of surfaces in space, and so on to higher dimensions.

Let $f(x,y,\alpha) = 0$ is a given curve with parameter α, then to evaluate envelope which is the equation of the curve including $f(x,y, \alpha) = 0$, we have to follow these steps

- Obtain $\frac{\partial}{\partial \alpha} f(x,y,\alpha) = 0$, from which we can get

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$$\alpha = g(x,y)$$

- Substitute $\alpha = g(x,y)$ in f $(x,y, \alpha) = 0$ such that the envelope is f $(x,y, g(x,y)) = 0$.

Example 7

Find the envelope of the following curves:

1- $x\cos\alpha + y\sin\alpha = P$, α is the parameter .

2- $(x- c)^2 + y^2 = 2c$

Solution:

1- $\dfrac{\partial}{\partial\alpha}(x\cos\alpha + y\sin\alpha = P)$, therefore $-x\sin\alpha + y\cos\alpha = 0$,

thus $\tan\alpha = y/x$, so $\cos\alpha = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\alpha = \dfrac{y}{\sqrt{x^2+y^2}}$, hence

envelope is $x^2 + y^2 = P^2$.

2- $\dfrac{\partial}{\partial c}((x- c)^2 + y^2 = 2c)$, therefore $2(x-c) = 2$, thus $x-1 = c$,

hence envelope is $1 + y^2 = 2(x-1) \Rightarrow 2x-3 = y^2$.

Problems

Find envelope of the following functions

1) $(x-\cos\alpha)^2 + (y-\sin\alpha)^2 = \rho$

2) $(x-\alpha)^2 + y^2 = 4$

3) $y = \alpha x + 1/\alpha$

4) $2c\,x + (y-c)^2 = 2y$

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Maxima and Minima

The problem of determining the maximum or minimum of function is encountered in geometry, mechanics, physics, and other fields, and was one of the motivating factors in the development of the calculus in the seventeenth century. Let us recall the procedure for the case of a function of one variable $y=f(x)$. First, we determine points x_c where $f'(x_c)=0$. These points are called critical points. At critical points the tangent line is horizontal. The second derivative test is employed to determine if a critical point is a relative

maximum or a relative minimum. If $f``(x_c) > 0$, then $x_c$ is a relative minimum. If $f``(x_c) < 0$, then x_c is a maximum. If $f``(x_c)=0$, then the test gives no information. The notions of critical points and the second derivative test carry over to functions of two variables. Let $z = f(x,y)$ and the critical points are points in the xy-plane where the tangent plane is horizontal. The tangent plane is horizontal if its normal vector points in the z direction. Hence, critical points are solutions of the equations because horizontal planes have normal vector parallel to z-axis. The two equations $f_x = 0$, $f_y = 0$ must be solved simultaneously to obtain the critical points (x_c , y_c) and by applying the second derivative test so that we get f_{xx}, f_{yy}, f_{xy}. If $\Delta(x_c ,y_c) = f_{xx}(x_c ,y_c) f_{yy}(x_c ,y_c) - [f_{xy}(x_c ,y_c)]^2 > 0$, then at (x_c , y_c) the function $z = f(x,y)$ is said to have maximum or minimum value, if f_{xx} $f_{yy} > 0$, then $z = f(x,y)$ is said to have minimum value while if f_{xx} $f_{yy} < 0$, then $z = f(x,y)$ is said to have maximum value but if $\Delta(x_c ,y_c) = f_{xx}(x_c ,y_c)$ $f_{yy}(x_c ,y_c) - [f_{xy}(x_c ,y_c)]^2 < 0$, then (x_c , y_c) is a saddle point and if $\Delta(x_c ,y_c) = 0$, then we have no decision.

Example 8

Determine the critical points and locate any relative minima, maxima and saddle points of function f defined by:

$$f(x,y)=2x^2 + 2xy + 2y^2 - 6x$$

Solution:

Find the first partial derivatives f_x and f_y such that:

$$f_x(x,y) = 4x + 2y - 6 \ , \ f_y(x,y) = 2x + 4y$$

The critical points satisfy the equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$ simultaneously, hence $4x+2y-6=0$, $2x+4y=0$.

The above system of equations has one solution at the point (2,-1). We now need to find the second order partial derivatives $f_{xx}(x,y)$, $f_{yy}(x,y)$ & $f_{xy}(x,y)$ such that $f_{xx}(x,y)= 4$, $f_{yy}(x,y) = 4$, $f_{xy}(x,y) = 2$. We now need to find Δ defined above so that $\Delta = f_{xx}(2,-1)f_{yy}(2,-1)-f_{xy}^2(2,-1)=(4)(4) -2^2 = 12$

Since Δ is positive and $f_{xx}(2,-1)$ is also positive, according to the above theorem function f has a local minimum at (2,-1). The 3-Dimensional graph of function f given above

shows that f has a local minimum at the point $(2,-1,f(2,-1)) = (2,-1,-6)$.

Example 9

Determine the critical points and locate any relative minima, maxima and saddle points of function f defined by

$$f(x , y) = 2x^2 - 4xy + y^4 + 2$$

Solution:

Find the first partial derivatives f_x and f_y such that:

$$f_x(x,y) = 4x - 4y \ , \ f_y(x,y) = - 4x + 4y^3$$

Determine the critical points by solving the equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$ simultaneously, such that: $4x - 4y = 0$ and $- 4x + 4y^3 = 0$.

The first equation gives $x = y$. Substitute x by y in the equation $- 4x + 4y^3 = 0$ to obtain $- 4y + 4y^3 = 0$, therefore $4y (-1 + y^2) = 0$, hence $y = 0$, $y = 1$ and $y = -1$. We now use the equation $x = y$ to find the critical points such that (0, 0), (1 , 1) and (-1 , -1) are the critical points .

Determine the second order partial derivatives such that:

$$f_{xx}(x,y) = 4, \ \ f_{yy}(x,y) = 12y^2, \ \ f_{xy}(x,y) = -4$$

We now use a table to study the signs of Δ and $f_{xx}(a,b)$ and use the above theorem to decide on whether a given critical point is a saddle point, relative maximum or minimum.

Critical point	(0,0)	(1,1)	(-1,1)
f_{xx}	4	4	4
f_{yy}	0	12	12
f_{xy}	-4	-4	-4
Δ	-16	32	32
	Saddle	Relative minimum	Relative minimum

Example 10

Determine the critical points and locate any relative minima, maxima and saddle points of function f defined by

$$f(x, y) = -x^4 - y^4 + 4xy.$$

Solution:

First partial derivatives f_x and f_y are given by

$$f_x(x,y) = -4x^3 + 4y \,,\, f_y(x,y) = -4y^3 + 4x$$

We now solve the equations $f_y(x,y) = 0$ and $f_x(x,y) = 0$ to find the critical points such that $-4x^3 + 4y = 0$, $-4y^3 + 4x = 0$. The first equation gives $y = x^3$. Combined with the second equation, we obtain $-4(x^3)^3 + 4x = 0$ which may be written as $x(x^4 - 1)(x^4 + 1) = 0$, so that $x = 0$, -1 and 1. We now use the equation $y = x^3$ to find the critical points $(0,0)$, $(1,1)$ and $(-1,-1)$. We now determine the second order partial derivatives such that $f_{xx}(x,y) = -12x^2$, $f_{yy}(x,y) = -12y^2$, $f_{xy}(x,y) = 4$. The table below shows the signs of Δ and $f_{xx}(a,b)$. Then the above theorem is used to decide on what type of critical points it is.

Critical point	(0,0)	(1,1)	(-1,1)
f_{xx}	0	-12	-12
f_{yy}	0	-12	-12
f_{xy}	4	4	4
Δ	-16	128	128
	Saddle	Relative max.	Relative max.

Lagrange multiplier (conditional extrema)

Let f(x,y), g(x,y) are functions in two variables with continuous partial derivatives such that g(x,y) = c is the constraint equation and f(x,y) has an extreme value at (x_c, y_c) which satisfy g(x , y) = 0.

$$If \nabla g(x_c, y_c) = \frac{\partial}{\partial x} g(x,y)i + \frac{\partial}{\partial y} g(x,y), \ j \neq 0 \text{ at } x = x_c, y = y_c,$$

then there exist a number λ such that:

$$\nabla f(x_c, y_c) = \lambda \nabla g(x_c, y_c) ,$$

λ is called Lagrange multiplier, therefore

$$\frac{\partial}{\partial x} f(x,y)i + \frac{\partial}{\partial y} f(x,y)j = \lambda \ [\frac{\partial}{\partial x} g(x,y)i + \frac{\partial}{\partial y} g(x,y)j \ ]$$

Thus

$$\frac{\partial}{\partial x} f(x,y) = \lambda \frac{\partial}{\partial x} g(x,y), \ \frac{\partial}{\partial y} f(x,y) = \lambda \frac{\partial}{\partial y} g(x,y) \ \& \ g(x,y) = c$$

By solving the 3 equations, we get extreme points. This method is called Lagrange multipliers.

Example 11

Find the dimensions of the box with largest volume if the total surface area is 64 cm^2.

Solution:

Before we start the process here note that we also saw a way to solve this kind of problem in Calculus I, except in those problems we required a condition that related one of the sides of the box to the other sides so that we could get down to a volume and surface area function that only involved two variables. We no longer need this condition for these problems.

Now, let's get on to solving the problem. We first need to identify the function that we're going to optimize as well as the constraint. Let's set the length of the box to be x, the width of the box to be y and the height of the box to be z. Let's also note that because we're dealing with the dimensions of a box it is safe to assume that x, y, and z are all positive quantities. We want to find the largest volume

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and so the function that we want to optimize is given by:

$$f(x,y,z) = xyz, \quad 2(xy+xz+yz) = 64 \Rightarrow xy + xz + yz = 32$$

Note that we divided the constraint by 2 to simplify the equation a little. Also, we get the function $g(x,y,z)$ from this $g(x,y,z) = xy+xz+yz$, thus we will obtain four equations such that solve.

$f_x = \lambda g_x \Rightarrow yz = \lambda (y+z)$ …….. *

$f_y = \lambda g_y \Rightarrow xz = \lambda (x+z)$ ………**

$f_z = \lambda g_z \Rightarrow xy = \lambda(x+y)$ ……… ***

$xy + xz + yz = 32$ …………….. ****

Solve the first three equations, we will get $x = y = z$, substitute in (****), thus $x^2 = 32/3$, therefore $x = y = z = \pm\sqrt{32/3}$.

Example 12

Find the maximum and minimum of f(x,y) = 5x-3y subject to the constraint $x^2 + y^2 = 136$.

Solution:

This one is going to be a little easier than the previous one since it only has two variables. Also, note that it's clear from the constraint that region of possible solutions lies on a disk of radius $\sqrt{136}$ which is a closed and bounded region and hence by the Extreme Value Theorem we know that a minimum and maximum value must exist. Here is the system that we need to solve:

$f_x = \lambda\, g_x \Rightarrow 5 = \lambda\,(2x)$……..(#)

$f_y = \lambda\, g_y \Rightarrow -3 = \lambda\,(2y)$…….(##)

$x^2 + y^2 = 136$………………(###)

Notice that, as with the last example, we can't have $\lambda = 0$ since that would not satisfy the first two equations. So, since $\lambda \neq 0$, we can solve the first two equations for x and y

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respectively by substituting $x = \dfrac{5}{2\lambda}$, $y = \dfrac{-3}{2\lambda}$ in (###), we

will get $\dfrac{25}{4\lambda^2} + \dfrac{9}{4\lambda^2} = \dfrac{17}{2\lambda^2} = 136$, thus $\lambda = \pm\dfrac{1}{4}$, at $\lambda = 1/4$,

therefore x = 10, y = -6 and at λ= -1/4, thus x = -10, y = 6,

therefore f(-10,6) = -68 (min.), f(10,-6) = 68 (max.).

Example 13

Find the maximum and minimum of f(x,y) = xyz subject to

the constraint x + y + z = 1, assume x, y, z > 0.

Solution:

First note that our constraint is a sum of three positive or

zero number and it must be 1. Therefore it is clear that our

solution will fall in the range 0< x, y, z < 1 Therefore the

solution must lie in a closed and bounded region and so by

the Extreme Value Theorem we know that a minimum and

maximum value must exist. Here is the system that we need

to solve:

$f_x = \lambda\, g_x \Rightarrow yz = \lambda$…………(i)

$f_y = \lambda\, g_y \Rightarrow xz = \lambda$…………...(ii)

$f_z = \lambda\, g_z \Rightarrow xy = \lambda$…………(iii)

$x + y + z = 1$…………………...(iv)

From the above equations, $yz = xz \Rightarrow z(x-y) = 0$ and $xz = xy \Rightarrow x(y-z)=0$, thus $z = 0$ or $x = y$, if $z = 0$, therefore $\lambda = 0$, thus $xy = 0$, hence either $x = 0$ or $y = 0$.

At $z = 0$, $x = 0$, therefore $y = 1$, or at $z = 0$, $y = 0$, thus $x = 1$ and so we've got two possible solutions $(0, 1, 0)$ & $(1, 0, 0)$.

If $x = y$, therefore $x = y = z = 1/3$ and if $x = y = 0$, therefore $z = 1$, so we've got two possible solutions $(0,0,1)$ and $(1/3,1/3,1/3)$, hence $f(0,0,1) = f(0,1,0) = f(1,0,0) = 0$ (min.) and $f(1/3,1/3,1/3) = 1/27$ (max.)

Problems

1) Find all relative extrema and saddle points for

$f(x, y) = 3x^2 + y^2 + 9x-4y+6$

2) Find all relative extrema and saddle points for

$f(x, y) = x^3 - y^2 - yx^2$

3) A retail outlet sells two different telephone answering machines. The demand functions are $Q_A = 100 - 2x+y$ and $Q_S = 90 + x - 1.5y$, where Q_A is the quantity demanded of brand A and Q_S is the quantity demanded of brand S when the prices charged are x dollars for brand A and y dollars for brand S. Determine the prices that would maximize total revenue from sales of both brands of answering machines. How many of each would be sold at those prices?

4) Maximize $f(x,y) = x+y$ subject to the constraint $x^2+y^2 = 1$.

5) Find the maximum and minimum of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \le 4$.

6) Find the maximum and minimum of $f(x,y) = x^2 + y^2$

subject to the constraint $x^2+xy+ y^2= 31$.

7) Find the maximum and minimum of $f(x,y)= yx^2+2y+ y^3$ subject to the constraint $xy-1 =0$.

8) Find maximum product of positive 3 numbers whose sum equal S.

9) An open box with volume 12 m^3, find the dimensions of the box to obtain maximum area.

10) Expand $f(x,y) = xy \sin(xy)$ in Taylor Maclaurin series.

11) Find envelope of the function $(x-\alpha)^2+(y-\alpha)^2 = P$, α is the parameter.

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